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Percolation in Strongly Correlated Systems: The Massless Gaussian Field

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We derive a number of new results for correlated nearest neighbor site percolation on Z^d. We show in particular that in three dimensions the strongly correlated massless harmonic crystal, i.e., the Gaussian random field with mean zero and covariance $-\Delta$, has a nontrivial percolation behavior: sites on which $S_x \geqslant h$ percolate if and only if $h < h_c$ with $0 \leqslant h_c < \infty$. This provides the first rigorous example of a percolation transition in a system with infinite susceptibility.

KEY WORDS: Percolation; weak and strong correlation; symmetry breaking; massless harmonic crystal.

1. INTRODUCTION

Percolation is a fascinating problem relevant for a variety of systems. Combining notions of geometry and randomness, it is easy to formulate and, in general, rather hard to analyze.[1–6] In this paper we try to answer some questions in the theory of percolation for "strongly coupled" systems.

Consider a random field, i.e., a set $\{S_x; x \in Z^d\}$ of random variables, with values in R. The joint distribution P of these variables will in the concrete examples discussed in this paper always be a Gibbs state of some statistical mechanical system. Define, for each $h \in R$, the random set

$$E(h) \equiv \{x \in Z^d: S_x \geqslant h\}$$

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sometimes called the excursion set. We ask whether this set $E(h)$ contains infinite connected components.

We can translate this question into the usual percolation problem by defining the occupation variables

$$\begin{aligned} \rho_x(h) &\equiv 1 \quad \text{if} \quad S_x \geqslant h \\ &\equiv 0 \quad \text{if} \quad S_x < h \end{aligned} \tag{1.1}$$

The probability measure $\tilde{P}$ on these variables $\{\rho_x(h)\}$ is the one induced by P. The question is then, do typical configurations $\tilde{S}$ contain infinite connected sets of occupied sites? In contrast with the independent case (Bernoulli site percolation), where the only relevant parameter is the density

$$\text{Prob}(x \text{ is occupied}) = \langle \rho_x(h) \rangle \equiv p(h) \tag{1.2}$$

the complete distribution $\tilde{P}$ will be relevant in general. The motivation of the present work is to understand better how interactions influence percolation (see also Ref. 7).

We define, as usual, the *percolation probability*: $P_\infty(h) \equiv$ Prob(the origin belongs to an infinite connected set of occupied sites). Clearly $P_\infty(h)$ is nonincreasing in h and we can define the critical level

$$h_c \equiv \text{Sup}\{h \in R \colon P_\infty(h) > 0\} \tag{1.3}$$

above which there is no percolation and below which there is percolation. The associated critical density is $\langle \rho_x(h_c) \rangle = p(h_c) \equiv p_c$. We say that there is a *percolation transition* if $|h_c| < \infty$ or, equivalently, $0 < p_c < 1$,

The problem was considered in general by Molchanov and Stepanov,[8] who constructed in arbitrary dimensions examples of fields with good symmetry and ergodic properties for which there is no percolation transition ($h_c = \pm\infty$). While their examples are somewhat artificial, they raise questions about the intuitive feeling that all "natural" fields, in $d > 1$, should have a nontrivial critical density.

Molchanov and Stepanov also gave a criterion implying the presence of a percolation transition. Intuitively, their criterion should apply to "weakly" correlated random fields. While it is not always easy to check whether a given system satisfies their criterion (see, however, Theorem 1), it is interesting to consider examples where it certainly fails. One such example is the massless harmonic crystal with mean zero, i.e.,

$$\langle S_x \rangle = 0$$

Its correlations (for $d > 2$) decay as

$$\langle S_x S_y \rangle \approx 1/|x-y|^{d-2}$$

and in particular, the "susceptibility"

$$\chi = \sum_{x \in Z^d} \langle S_0 S_x \rangle = \infty \tag{1.4}$$

This shows that we are dealing with a strongly correlated random field. While it is rather easy to show (using an idea of Russo) that $h_c \geqslant 0$ (i.e., $p_c \leqslant 1/2$), it is less obvious that $h_c < \infty$. It is *a priori* conceivable that, due to strong correlations in the system, infinite connected sets exist where $S_x \geqslant h$, no matter how large h is. However, using ideas of potential theory, we show that this is not the case for $d = 3$, i.e., $h_c < \infty$.

In Section 3 we state precisely our results. First we show that for distributions satisfying the GHS inequality, the Molchanov–Stepanov criterion[8] is satisfied whenever the "A-susceptibility"

$$\chi_A \equiv \frac{1}{A} \sum_{x, y \in A} \langle S_x S_y \rangle \tag{1.5}$$

is uniformly bounded in A. Next, we combine an idea of Russo[6] and correlation inequalities to show the presence of percolation for one- and two-component spin systems in their multiple-phase region. Finally, we state our results for the harmonic crystal. Section 4 is devoted to the proofs of these results and Section 5 to a discussion of some possible extensions.

2. NOTATIONS AND DEFINITIONS

We consider a cubic lattice Z^d in $d > 1$ dimensions. Its elements are the sites $x = (x_1, \dots, x_d)$ with $x_i \in Z$. For a point $x \in Z^d$, the length of x (=its distance from the origin 0 of the lattice) is $|x| \equiv \sum_1^d |x_i|$, where x_i is the ith component of x. Two sites x, $y \in Z^d$ are *nearest neighbors* if their distance $|x - y| = 1$.

If $K \subset Z^d$, we define the *outer boundary* of K,

$$\partial K \equiv \{x \in Z^d \backslash K : \exists y \in K, |x - y| = 1\} \tag{2.1}$$

and the *inner boundary* of K,

$$\delta K \equiv \{x \in K : \exists y \in Z^d \backslash K, |x - y| = 1\} \tag{2.2}$$

We put $\bar{K} = \partial K \cup K$. The volume $|K|$ of K is its cardinality, i.e., the number of sites $x \in K$. The set K is finite if $|K| < \infty$. A set $K \subset Z^d$ is *connected* if any two points x, $y \in K$ can be joined by a path of nearest neighbors in K.

The configuration space Ω is the set of sequences $\tilde{S} = (S_x)_{x \in Z^d}$ of random variables $S_x \in R$. The restriction of $\tilde{S} \in \Omega$ to a region $K \subset Z^d$ is written $\tilde{S}_K = (S_x)_{x \in K}$ and is not to be confused with the sum $S_K \equiv \sum_K S_x$. Given a configuration $\tilde{S}$, a connected set $K \subset Z^d$ is a *cluster* if $\rho_x = 1$, for all $x \in K$, and $\rho_x = 0$, for all $x \in \partial K$, where ρ_x was defined in (1.1).

The measure P is defined on the σ-algebra of Borel sets inherited by the product topology in $\Omega \equiv R^{Z^d}$. The indicator function of an event E, i.e., a particular subset of Ω, is denoted by $I(E)$. The probability of E is $P(E) = \text{Prob}(E) = \langle I(E) \rangle$, where $\langle \cdot \rangle$ is the expectation value of $\cdot$ with respect to the measure P. We assume, unless otherwise stated, that $\langle S_x \rangle = 0$.

3. RESULTS

3.1. Weakly Correlated Random Fields: The Molchanov–Stepanov Criterion

Molchanov and Stepanov[8] prove the following general criterion for percolation: if the probability of a set being occupied (empty) decays exponentially with the volume of that set, and the rate of decay is large enough, then percolation does not (does) occur. More precisely:

Criterion[8]. Let $A \subset Z^d$ be an arbitrary but finite connected set. Suppose that

$$\langle \rho_A(h) \rangle \equiv \text{Prob}(S_x \geqslant h, \forall x \in A) \leqslant c \exp(-\alpha |A|) \tag{3.1}$$

where α and c are independent of A, with $\alpha = \alpha(h) \uparrow +\infty$ as $h \uparrow +\infty$, and $c = c(h) < \infty$ (not necessarily uniformly bounded in h). Then $h_c < +\infty$. Similarly, if

$$\text{Prob}(S_x < h, \forall x \in A) \leqslant c' \exp(-\alpha' |A|)$$

with c', α' as in (3.1), then $h_c > -\infty$.

The proof of this criterion is similar to the "Peierls argument" for independent percolation.

To see when this criterion is satisfied, consider first the situation where the conditional probability,

$$\text{Prob}(S_x \geqslant h \,|\, \text{any configuration in } Z^d \backslash x) \leqslant e^{-\alpha} \tag{3.2}$$

with $\alpha = \alpha(h)$ large uniformly in the configuration outside x. Then clearly (3.1) holds. However, (3.2) is usually difficult to check, except when the probability of $S_x \geqslant h$ is small and the events $S_x \geqslant h$ for different sites x are weakly correlated. Examples of such cases include (besides independent percolation) infinite-volume Gibbs states with respect to a superstable and regular interaction (see Ref. 9 for the definitions). In that case the probability estimates of Ruelle are independent of the shape of the set A.

Another way to verify (3.1) is to observe that

$$\mathrm{Prob}(S_x \geqslant h, \forall x \in A) \leqslant \mathrm{Prob}(S_A \geqslant h\,|A|) \tag{3.3}$$

which allows the use of large-deviation estimates when available for arbitrary connected sets. Relation (3.3) can also be combined with GHS (or Gaussian[10]) inequalities to obtain the following result: define for $t \geqslant 0$ the expectation values

$$\langle \cdot \rangle^t = [\langle \exp(tS_A) \rangle]^{-1} \langle \cdot \exp(tS_A) \rangle$$

Theorem 1. Let $M_x^t = \langle S_x \rangle^t$. If the GHS inequality

$$\langle (S_x - M_x^t)(S_y - M_y^t)(S_z - M_z^t) \rangle^t \leqslant 0 \tag{3.4}$$

holds for all $x, y, z \in A$, and, if the A-susceptibility

$$\chi_A = \frac{1}{|A|} \sum_{x, y \in A} \langle S_x S_y \rangle$$

is bounded uniformly in A, then $h_c < +\infty$ (or $p_c > 0$).

Theorem 1 shows that boundedness of the susceptibility may be taken, in our context, as an indicator that the random field is weakly correlated.

3.2. A Generalization of Russo's Argument

In Lemma 1 of Ref. 6, Russo characterized the pure phases of the nearest neighbor Ising model (in $d = 2$) in terms of the existence of infinite clusters: if a Gibbs state has no infinite $(+)$ cluster, with probability one, then it is the $(-)$ state. This implies percolation [of $(+)$ spins in the $(+)$ state, for all temperatures below T_c] in a situation where the arguments of the previous subsection do not apply: a Peierls-type argument works only at low enough temperature.

Russo's argument should extend to many models where several phases coexist. Indeed, phase coexistence means that the spin at the origin "feels" the boundary conditions "at infinity." However, if no percolation takes place, then the spin at the origin will be screened off from infinity by the occurrence (with probability one) of some surface surrounding the origin where all sites are empty.

We give here an abstract but somewhat weaker version of Russo's argument, which applies to more general models in their multiple-phase region.

We first define $C \subset Z^d$ as a *contour surrounding the origin* if there exists a finite connected set K containing the origin such that $C = \partial K$, or $C = \{0\}$, in which case we still write $C = \partial K$, $K = \varnothing$. Remark that K is uniquely defined: the *interior* of $C = \mathrm{Int}\, C = K$, if $C = \partial K$. For C any contour surrounding the origin, let $C_<$ denote the event that all sites on C are empty:

$$C_< \equiv \{\tilde{S} \in \Omega: \rho_x(h) = 0, \forall x \in C\} \tag{3.5}$$

We call C a $<$-contour if $C_<$ occurs.

Suppose that the random field has the one-step Markov property,[11] e.g., P is a Gibbs state for a Hamiltonian with nearest neighbor interactions. Then the contour C splits the interior of C from the exterior of C: the probability measure in the interior of C is completely determined by the configurations $\tilde{S}_C$ on the contour C, independent of the state in the exterior of C. Let $\langle \cdot \rangle_{\tilde{S}_C}$ be the expectation value with respect to the measure in the interior of the contour C, obtained from the original measure P by imposing the boundary condition $S_x, x \in C$, on C, i.e., the probability distribution, conditioned on $\tilde{S}_C$.

Theorem 2. Let P satisfy the one-step Markov property. If there exists a bounded function $f: R \to R$, and a $\delta \in R$ such that the unconditioned expectation satisfies

$$\langle f(S_0) \rangle > \delta \tag{3.6}$$

while the expectation conditioned on an empty contour, $S_x < h$ on C, (3.5), satisfies

$$\langle f(S_0) \rangle_{\tilde{S}_C} \leqslant \delta, \qquad \text{for all contours } C \text{ surrounding the origin} \tag{3.7}$$

then, $P_\infty(h) > 0$.

Remark. This result is weaker than Russo's result: it only gives percolation of $S_x \geqslant h$, but otherwise does not imply anything about the state $\langle \cdot \rangle$. However, in order to apply it, we are free to pick any function f satisfying (3.6) and (3.7). The following corollaries will be proven in Section 4.

Corollary 1. Take a general one-component spin model with nearest neighbor ferromagnetic interactions,

$$-H = \sum_{\langle xy \rangle} S_x S_y; \qquad S_x \in R \tag{3.8}$$

and an even single-spin measure $\lambda(S_x)$ (suitably decaying for large S_x so that the model is well defined).[9] There will be percolation of sites x where

$S_x \geqslant 0$ in any Gibbs state where $\langle \text{sign } S_0 \rangle > 0$. Such states always occur at low enough temperatures, in $d \geqslant 2$, provided $\lambda(S) \neq \delta(S)$, by Wells' inequality.[12]

Corollary 2. Consider now one-component models that are invariant under shifts

$$-H = \sum_{\langle xy \rangle} \psi(S_x - S_y) \tag{3.9}$$

where ψ is an even and convex function. $S_x \in R$ or Z and the single-spin measure is flat; $\psi(t) = |t|$ and $S_x \in Z$ is the SOS model; $\psi(t) = \frac{1}{2}t^2$ and $S_x \in R$ is the harmonic crystal (Section 3.3), etc.

Let P be a Gibbs state with zero boundary conditions. $d \geqslant 3$, $\psi(t) = \alpha t^2 + v(t)$, where $\alpha > 0$ and v is convex, are sufficient conditions for the existence of such a state.[13] There will be percolation of sites x where $S_x \geqslant -l$, for any $l > 0$.

Corollary 3. Consider a two-component model:

$$-H = \sum_{\langle xy \rangle} \mathbf{S}_x \cdot \mathbf{S}_y \tag{3.10}$$

where $\mathbf{S}_x \in R^2$, $\mathbf{S}_x = (r_x \cos \phi_x, r_x \sin \phi_x)$, and a rotation-invariant single spin measure $\lambda(\mathbf{S}_x) = \lambda(r_x)$. There is percolation of sites x where $\cos \phi_x \geqslant 0$, in any Gibbs state where $\langle \cos \phi_0 \rangle > 0$. These occur at low temperatures if $d \geqslant 3$.[12,14]

3.3. Strongly Correlated Gaussian Fields: The Massless Harmonic Crystal

The massless harmonic crystal in Z^d $(d \geqslant 3)$ is the Gaussian random field with mean zero.

$$\langle S_x \rangle = 0$$

and covariances

$$\langle S_x S_y \rangle = c_{xy} = -(\Delta^{-1})_{xy} \tag{3.11}$$

$-\Delta$ is the lattice Laplacian on $l_2(Z^d)$, i.e.,

$$-\Delta f(x) \equiv 2d \left[f(x) - \frac{1}{2d} \sum_{|x-y|=1} f(y) \right] \tag{3.12}$$

for a function $f(x)$, $x \in Z^d$.

For $d=3$ there are strictly positive constants α_1 and α_2 such that

$$\alpha_1/|x-y| \leqslant c_{xy} \leqslant \alpha_2/|x-y|, \qquad |x-y| \neq 0 \tag{3.13}$$

This field corresponds to the infinite-volume Gibbs state with respect to the formal Hamiltonian,

$$H(\tilde{S}) = \frac{1}{2} \sum_{nn} (S_x - S_y)^2 \tag{3.14}$$

where the sum is over pairs of nearest neighbors. The simpler massive case is obtained by adding a term

$$\frac{m^2}{2} \sum_x S_x^2$$

to (3.14). Since any sum of jointly Gaussian random variables is Gaussian, one has the following bound[15]: for $h>0$,

$$\operatorname{Prob}(S_x \geqslant h, \forall x \in \Lambda) \leqslant \operatorname{Prob}(S_\Lambda \geqslant h|\Lambda|) \leqslant \left(\frac{\chi_\Lambda}{2\pi|\Lambda|}\right)^{1/2} \frac{1}{h} \exp\left(-\frac{h^2}{2}\frac{|\Lambda|}{\chi_\Lambda}\right) \tag{3.15}$$

with χ_Λ given by (1.5). Now, for $m>0$, the Λ-susceptibility χ_Λ is bounded uniformly in Λ, so one can satisfy the criterion (3.1) of Section 3.1 (see also Theorem 2.4 in Ref. 8). Thus, $-\infty < h_c < \infty$ for any $m>0$.

In the massless case, however, it is not *a priori* obvious that $h_c < \infty$: let $\Lambda = \{1, 2, ..., n\}$ be a line segment of length n in Z^3; then, using (3.13), one has

$$\chi_\Lambda \approx \frac{1}{n} \sum_{\substack{x,y=1 \\ x\neq y}}^{n} \frac{1}{|x-y|} \approx \log n \tag{3.16}$$

Therefore, the bound (3.15) is useless. Moreover, it follows from Lemma 1 in Ref. 16 that the Molchanov–Stepanov criterion is not satisfied in the massless case. Nevertheless, by a different method explained below, we can prove the following.

Theorem 3. For $d=3$, $0 \leqslant h_c < \infty$.

The lower bound is an immediate consequence of Corollary 2 in Section 3.2. The upper bound is more subtle. We indicate here the main ideas of the proof. First we observe that the "average magnetization" in a box Λ, $|\Lambda| = L^3$, defined by

$$\frac{1}{|\Lambda|} S_\Lambda \equiv \frac{1}{|\Lambda|} \sum S_x \tag{3.17}$$

is zero up to very small fluctuations (of order $1/L$). We want to show that this will contradict the existence of an infinite cluster where $S_x \geqslant h \gg 1$. Indeed, *fix* an infinite cluster C and condition on the event that $S_x \geqslant h$, for all $x \in C$, or, taking the "worst case," that $S_x = h$ on C. Then, it follows from potential theory (to which the correlations in the massless crystal are intimately related; see Ref. 17 and Lemma 2 in Section 4) that

$$\langle S_x \mid S = h \text{ on } C \rangle = h, \qquad \text{for all } x \text{ in } Z^3 \tag{3.18}$$

This only uses that C is an infinite, connected set in $d = 3$. The heuristic argument behind (3.18) is the following: first we observe that for a given C,

$$\langle S_x \mid S = h \text{ on } C \rangle = h \text{ Prob(a random walker starting at } x \text{ will visit } C)$$

Now, in $d = 3$, an infinite, connected set will be visited with probability one (easily proven for C a coordinate axis, where hitting C is equivalent to visiting the origin for a two-dimensional random walk, which is well known to occur with probability one.[17] The extension to general sets C is due to Itô and McKean[18,17] and is the content of Lemma 2 in Section 4). Given this fact, we would like to argue that if there is *any* C where $S_x \geqslant h$, it would tend to lift the typical level of the other spins. In particular, it "pushes up" the S_y, $y \in \Lambda$, and this would contradict the fact that $(1/\Lambda)$ $S_\Lambda \approx 0$. More precisely, if we can show that

$$\langle (1/\Lambda)\, S_\Lambda \mid \exists \text{ infinite connected } C \rangle = h$$

then Prob($\exists$ infinite connected C) < 1, and $P_\infty(h) = 0$.

Of course this argument is deceitfully simple. Actually, if it worked, it would imply absence of percolation for all $h > 0$, i.e., $p_c = 1/2$, which we do not expect to be true.[19] The point is that the different events $\{S_x \geqslant h$, $\forall x \in C)$ for different C's are not disjoint. To obtain disjoint events, which is done in Lemma 1 (Section 4), one has to consider the largest set, containing the origin, on which $S_x \geqslant h$. However, this event implicitly contains the information that $S_y < h$ for y on the boundary of C. One has to use the additional fact that, if h is large enough, then these S_y, $y \in \partial C$, are still large because the harmonic crystal does not like to develop large gradients. This is essentially the content of Lemma 3 (Section 4). Putting the three lemma's together, one obtains a contradiction between the fact that $(1/|\Lambda|)$ $S_\Lambda \approx 0$ and the existence of an infinite, connected cluster of occupied sites, i.e., a proof of the upper bound in Theorem 3.

4. PROOFS

In this section, we shall usually not indicate explicitly the h dependence of various quantities.

Proof of Theorem 1. Let $f_A(t) \equiv (1/|A|) \log\langle \exp(tS_A)\rangle$. By a Chebyshev inequality we have for all $t \geqslant 0$,

$$\mathrm{Prob}(S_A \geqslant h|A|) \leqslant \exp\{|A| [-ht + f_A(t)]\} \tag{4.1}$$

Since $f'_A(0) = 0$, we get, by Taylor's theorem with remainder

$$f_A(t) = t^2 \int_0^1 (1-s) f''_A(st)\, ds$$

By (3.4), $f'_A(t)$ is concave on $[0, +\infty)$, so that

$$f''_A(st) \leqslant f''_A(0) = \chi_A$$

Hence, the upper bound in (4.1) becomes

$$\exp[|A| (-ht + \tfrac{1}{2} t^2 \chi_A)]$$

The conclusion follows from the percolation criteria (3.1) and (3.3), since χ_A is assumed to be uniformly bounded.

Proof of Theorem 2. For any family $C_1,..., C_n$ of contours surrounding the origin, one defines the maximal contour

$$\max\{C_1,..., C_n\} \equiv \partial\left(\bigcup_1^n \mathrm{Int}\, C_i\right)$$

Let Λ_N be a cube around the origin of size N. Let

$$\begin{aligned} C_<^{\max} \equiv \{\tilde{S} \in \Omega\colon & C \text{ is the maximal } <\text{-contour} \\ & \text{of } \tilde{S} \text{ surrounding the origin} \\ & \text{which is contained in } \Lambda_N\} \end{aligned}$$

and define

$$M \equiv M(\Lambda_N) \equiv \bigcup_{C\text{-in } \Lambda_N} C_<^{\max}$$

This last union runs over disjoint events and clearly any $\tilde{S} \in C_<$ for some $C \subset \Lambda_N$ also belongs to $M(\Lambda_N)$. Therefore, if $\tilde{S} \in \bar{M}$, the complement of M, there is no $<$-contour surrounding the origin, i.e., there exists a cluster connecting the origin to $\delta\Lambda_N$. We shall show that $\mathrm{Prob}(\bar{M}) > 0$ uniformly in N, which implies percolation.

Conditioning on $C_<^{\max}$ involves only the configuration on C and its exterior. In particular, for variables in the interior of C, this conditional expectation reduces to conditioning on the configurations in $\tilde{S}_C \in C_<$. The hypothesis of Theorem 2 states that there exists $f(S_0)$ such that

$$\langle f(S_0) \mid C_<^{\max} \rangle \leqslant \delta \tag{4.2}$$

and that

$$\langle f(S_0) \rangle = \sum_C \langle f(S_0) \mid C_<^{\max} \rangle \operatorname{Prob}(C_<^{\max}) + \langle f(S_0) \mid \bar{M} \rangle \operatorname{Prob}(\bar{M}) = \delta + \eta \tag{4.3}$$

with $\eta > 0$. We now observe that $\sum_C \operatorname{Prob}(C_<^{\max}) \leqslant 1$ since the $C_<^{\max}$ are disjoint events, so that, by using (4.2) and (4.3), we get the inequality

$$\delta + \eta \leqslant \delta + \langle f(S_0) \mid \bar{M} \rangle \operatorname{Prob}(\bar{M})$$
$$\delta + \eta \leqslant \delta + \text{const} \cdot \operatorname{Prob}(\bar{M})$$

where the positive const $\geqslant f(S)$, for all S, bounds the function f. This completes the proof.

Proof of the Corollaries. *Corollary* 1. Take $h = 0$, $\delta = 0$, and $f(S) = \text{sign } S$ in Theorem 2. By the FKG inequality,[20]

$$\langle f(S_0) \rangle_{\tilde{S}_C} \leqslant \langle f(S_0) \rangle_{\tilde{S}_C = 0} = 0$$

The last equality follows from symmetry. From Theorem 2, we thus get percolation of sites where $S_x \leqslant 0$ in any Gibbs state where $\langle \text{sign } S_0 \rangle > 0$.

Corollary 2. For any $l > 0$, take $h = -l$, $\delta = 0$, and $f(S) = \text{sign}(S + l)$ in Theorem 2. Then, by symmetry, $\langle f(S_0) \rangle = \operatorname{Prob}(S_0 \in [-l, l[) > 0$. This strict inequality can be proven using the DLR equality.[9] The measure satisfies the FKG inequalities by the convexity of ψ,[21] and is invariant, up to a change in the boundary condition, under a uniform shift of all the spins S_x. Hence, after a change of variables $S' = S + l$, the same argument as in Corollary 1 above applies: if $S_x \leqslant -l$ on C, then $\langle f(S_0) \rangle_{\tilde{S}_C} \leqslant 0$. By Theorem 2, there is percolation of sites x where $S_x \geqslant -l$.

Corollary 3. Take $h = 0$, $\delta = 0$, and $f(S_0) = \cos \phi_0$ in Theorem 2 and use the correlation inequality

$$\langle \cos \phi_0 \rangle_{\tilde{S}_C} \leqslant 0 \tag{4.4}$$

for any boundary condition $\tilde{S}_C$ where $\cos \phi_x \leqslant 0$, and $\sin \phi_x$ arbitrary. Theorem 2 then yields percolation of sites where $\cos \phi_x \geqslant 0$ in any Gibbs state where $\langle \cos \phi_0 \rangle > 0$.

To prove (4.4), first change all ϕ_x into $-(\phi_x-\pi)$. This leaves the interaction invariant and changes the boundary condition from $\cos\phi_x \leqslant 0$ to $\cos\phi_x \geqslant 0$, still leaving $\sin\phi_x$ unchanged, with an arbitrary sign. In these new variables we must show that $\langle\cos\phi_0\rangle_{\tilde{S}_C} \geqslant 0$. If all $\sin\phi_x$ on C were positive, this would just be the first Griffiths inequality for rotators.[22] However, $\langle\cos\phi_0\rangle_{\tilde{S}_C}$ with arbitrary fields in the $\sin\phi_x$ direction is larger than the one with all those fields replaced by their absolute value [inequality (A.4) in Ref. 22]. This proves (4.4).

Proof of Theorem 3. We start the proof by introducing some definitions and notation. Let V and Λ be cubes centered around the origin, $|V| \gg |\Lambda|$. Let $C(0) \equiv$ the cluster containing the origin and define the event

$$C_V \equiv \{\tilde{S} \in \Omega : C(0) \cap \delta V \neq \varnothing\} \tag{4.5}$$

Let F_V be the collection of sets B such that $B \subset V: 0 \in B$, B is connected, and $B \cap \delta V \neq \varnothing$.

Define for a particular $K \in F_V$ the event

$$E_K \equiv \{\tilde{S} \in \Omega : S_x \geqslant h, \forall x \in K \text{ and } S_x < h, \forall x \in \partial_V K\} \tag{4.6}$$

where

$$\partial_V K \equiv \partial K \cap V = \{x \in V \backslash K : \exists y \in K, |x-y| = 1\}$$

Lemma 1. (a) C_V is the disjoint union of the events E_K, i.e.,

$$C_V = \bigcup_{K \in F_V} E_K$$

and if $K, K' \in F_V$ and $K \neq K'$, then $E_K \cap E_{K'} = \varnothing$.

(b) $\text{Prob}(C_V) \geqslant P_\infty(h)$ for all V and $\text{Prob}(E_K) > 0$ for all $K \in F_V$, all finite V.

Proof. If $C(0)$ intersects the boundary of V, then the intersection of $C(0)$ with V contains a set $K \in F_V$ and E_K occurs. If E_K occurs for some $K \in F_V$ then K is a subset of $C(0)$ and C_V occurs. The events E_K are disjoint by definition. Part (b) is obvious.

Before continuing with the rest of the proof, we have to introduce (very briefly and incompletely) some elements of Newtonian potential theory on Z^d, i.e., electrostatics on the lattice. Details can be found in Spitzer's book.[17] It is here that the restriction to the harmonic crystal in Z^3 enters.

A function $f(x)$, $x \in Z^d$, is *harmonic* in a region $M \in Z^d$ of space if $-\Delta f(x) = 0$ for all $x \in M$. [Δ is defined in (3.12.] The (normalized) covariance c_{0x}/c_{00} for the harmonic crystal is a harmonic function for $x \neq 0$, which takes the value 1 for $x=0$ and decays to zero as $|x| \uparrow \infty$. More generally we can construct a function f_A that is identically one on a set $A \in Z^d$, harmonic off A, and decaying to zero at infinity. $f_A(x)$ is called the *eqilibrium potential* of A. It is also the probability for a simple random walk starting at position x to hit the region A. We have the following decomposition:

$$f_A(x) = \sum_{y \in A} c_{xy} e_y^A + q_A \tag{4.7}$$

For any finite, nonempty set A, q_A is zero and the $\{e_y^A;\, y \in A\}$ in (4.7) are the unique solution to the set of linear equations

$$\sum_{y \in A} c_{xy} e_y = 1 \qquad \text{for all} \quad x \in A \tag{4.8}$$

If we define the matrix C_A to be the covariance matrix (which is positive definite) restricted to the set A, then it is trivial that

$$e_y^A = \sum_{z \in A} (C_A^{-1})_{yz} \qquad \text{for all} \quad y \in A \tag{4.9}$$

is the solution of (4.8). For infinite A, q_A is either one or zero, depending only on the set A.

The $\{e_y^A\}$ are called the *equilibrium charges* of A, and f_A satisfies Poisson's equation:

$$-\Delta f_A(x) = e_x^A \qquad \text{for all} \quad x \in Z^d \tag{4.10}$$

From this we see that the charges are nonzero (and positive) only at the boundary of the region A, i.e.,

$$\begin{aligned} e_y^A &\neq 0, \qquad \text{if} \quad y \in \delta A \\ e_y^A &= 0, \qquad \text{if} \quad y \notin \delta A \end{aligned} \tag{4.11}$$

and actually,

$$e_y^{\delta A} = e_y^A \qquad \text{for all } y \tag{4.12}$$

(In the context of random walks, the charges on A correspond to escape probabilities, i.e., e_y^A is the probability of never visiting A after leaving position $y \in A$.) The total equilibrium charge of a finite set A is called the *capacity*, Cap(A), of the set A:

$$\sum_{x \in A} e_x^A \equiv \text{Cap}(A) \tag{4.13}$$

For infinite A, one defines $\mathrm{Cap}(A)=\infty$. It is well known[17] that the capacity is a monotone set function, i.e.,

$$\text{if} \quad A\subset B, \qquad \text{then} \quad \mathrm{Cap}(A)\leqslant \mathrm{Cap}(B) \tag{4.14}$$

The capacity depends on, besides the volume, also the "shape" of the set. It measures in a way the "thinness" of a set: a long, thin set has a far larger capacity to absorb charge than rounder bodies of the same volume. (In the probabilistic analog: a thin or sparse set A provides a better opportunity for the random walk to escape from A than other sets of the same cardinality.[17]) These considerations are reflected in the inequalities

$$M\geqslant \frac{|A|}{\mathrm{Cap}(A)}\geqslant m \tag{4.15}$$

where

$$m\equiv \operatorname*{Min}_{y\in A} \sum_{x\in A} c_{xy}, \qquad M\equiv \operatorname*{Max}_{y\in A} \sum_{x\in A} c_{xy}$$

This is a direct consequence of the definition (29) and (27). In other words, for sets A that are "homogeneous" enough ($m\approx \chi_A\approx M$), $\mathrm{Cap}(A)\approx |A|/\chi_A$.

The potential $f_A(x)$ is pointwise increasing as the set A grows in size. This is a consequence of the principle of domination for harmonic functions. It is also intuitively clear from the interpretation of the potential as a hitting probability (see above) that f_A depends on the shape of A. How "thick" an infinite set A has to be in order for $f_A(x)$ to be equal to 1, for all x, is obviously dimension-dependent. In $d=2$, a random walker returns to the origin of the lattice with probability one. Therefore, in $d=3$ we may expect that $f_A(x)=1$ for all x if A is an infinite, connected set, since it is already the case for a coordinate axis of Z^3. A precise criterion was given by Itô and McKean,[18] Spitzer,[17] and others. It goes under the name:

Wiener's Test (Itô, McKean).[18,17] Given an infinite set $A\subset Z^3$, let A_n denote the intersection of A with the spherical shell of points x such that $2^n\leqslant |x|\leqslant 2^{n+1}$. Then $f_A(x)=1$ for all x if and only if

$$T[A]\equiv \sum_{1}^{\infty} 2^{-n}\,\mathrm{Cap}(A_n)=+\infty \tag{4.16}$$

In other words, $T[A]$ is the correct "measure" of the influence of the "conductor" A on the value of the potential in distant regions of space.

Lemma 2 (Potential problem). (a) $\langle S_x|E_K\rangle$ is a harmonic function outside $\bar{K}$.

(b) For some $\mu > 0$ and for all A, we can take $V = V(A)$ large enough so that

$$\frac{1}{|A|} \sum_{x \in A} f_{\bar{K}}(x) \geqslant \mu \qquad \text{for all} \quad K \in F_V$$

Proof. (a) Let $x \in Z^d \backslash \bar{K}$. Using the DLR equation,[23] we have that

$$\langle S_x | E_K \rangle = \langle \langle S_x \rangle (S_y;\ y \text{ is n.n. to } x) | E_K \rangle$$

where

$$\langle S_x \rangle (S_y;\ y \text{ is n.n. to } x) = \frac{1}{2d} \sum_{|y-x|=1} S_y$$

is the expectation value of S_x if one specifies the values of its $2d$ nearest neighbors. This implies that

$$-\Delta \langle S_x | E_K \rangle = 0$$

(b) The proof of the Itô–McKean result[18] above implies that, as the set A grows (in the sense of inclusion) such that $T[A] \uparrow \infty$, then the function $f_A \uparrow 1$. Therefore, since the set A is a fixed and bounded region in Z^3, it is sufficient to show that $T[K]$ can be made arbitrary large for all $K \in F_V$ and V large enough. One has thus to verify condition (4.16) for an arbitrary infinite connected set A in $d = 3$. We will do this in two steps: first we reduce the volume of A and then we show that no set A is worse than a straight line. By the monotonicity property (4.14), $T[A] \geqslant T[a]$ where $a \subset A$ is obtained by keeping (in a nonunique but arbitrary fashion) for each $i = 0, 1, 2,...$ only one point in the intersection of A with the ith shell $=$ $\{y$: y is on the boundary of the cube of size $2i\}$. The volume $|a_n| = 2^n$, and by (4.15)

$$\operatorname{Cap}(a_n) \geqslant 2^n / M_n \tag{4.17}$$

where

$$M_n \equiv \operatorname*{Max}_{x \in a_n} \sum_{y \in a_n} c_{xy} \tag{4.18}$$

Fix $x \in a_n$. Then order $y \in a_n$, $y \neq x$, according to their distance from x. The kth point in that order is at a distance at least k from x. Thus, using the second inequality in (3.13), we get

$$M_n \leqslant \text{const} \cdot \sum_{k=1}^{2^n} \frac{1}{k} \leqslant \text{const}' \cdot n \tag{4.19}$$

The upper bound is what we would get for M_n if a_n was a straight line; see also (3.16). Combining the inequalities (4.17)–(4.19) we thus get that $T[A] \geqslant \text{const} \cdot \sum (1/n)$, hence the desired divergence.

Lemma 3 (Stability estimate). For $h < \infty$ large enough there is a constant $c > 0$ such that for all V large enough

$$\langle S_x \mid E_K \rangle \geqslant c \qquad \text{for all} \quad x \in \partial K, \quad \text{all} \quad K \in F_V$$

Proof. The boundary ∂K splits the space into two separate regions: K and $Z^d \backslash \bar{K}$. Therefore, since the interaction is nearest neighbor, and forgetting about the constraint E_K for the moment, the induced measure $P(dS_{\partial K})$ on $\{S_x; x \in \partial K\}$ is a product of two Gaussian measures:

$$P(dS_{\partial K}) = \frac{1}{N} dS_{\partial K} \exp\left[-\frac{1}{2} \left(\sum a_{xy} S_x S_y + \sum b_{xy} S_x S_y \right) \right]$$

where the sums run over $x, y \in \partial K$; $N =$ normalization (while its real value will change from place to place in the arguments below, we still use the same notation); $\exp(-\frac{1}{2} \sum a_{xy} S_x S_y)$ is what is obtained by "integrating out" the variables S_z for $z \in Z^3 \backslash \bar{K}$, and includes also the self-interaction of ∂K; and $\exp(-\frac{1}{2} \sum b_{xy} S_x S_y)$ is what is obtained by "integrating out" the variables S_z for $z \in K$.

The set $K = \text{Int}\, \partial K$ and the Hamiltonian (3.14) is invariant under shifts $S_x \to S_x + k$, for all x. Hence, (b_{xy}) is also shift-invariant:

$$\sum b_{xy} S_x S_y = \sum b_{xy} (S_x + k)(S_y + k), \qquad \text{for all} \quad k \in R \tag{4.20}$$

We observe that by (4.9) the sum

$$\sum_y (a_{xy} + b_{xy}) = e_x^{\bar{K}} = e_x^{\partial K} \equiv e_x$$

is exactly the charge at $x \in \partial K$ associated to the potential $f_{\bar{K}} = f_{\partial K}$, as defined in (4.7). The last equality is obtained by observing that both functions equal one on the set ∂K, vanish at infinity, and are both harmonic off ∂K (because $e_x^{\bar{K}} = 0$ if $x \in K$).

We rewrite

$$\begin{aligned}
\sum a_{xy} S_x S_y &= \sum a_{xy} (S_x + k)(S_y + k) - 2k \sum a_{xy} S_x - k^2 \sum a_{xy} \\
&= \sum a_{xy} (S_x + k)(S_y + k) - 2k \sum e_x S_x \\
&\quad + 2k \sum b_{xy} S_x - k^2 \sum e_x + k^2 \sum b_{xy} \\
&= \sum a_{xy} (S_x + k)(S_y + k) - 2 \sum k_x S_x - k^2 \, \text{Cap}(\partial K) \qquad (4.21)
\end{aligned}$$

In the last equality we used (4.20) and the definition (4.13) of capacity of a set, and we denote $k_x \equiv ke_x$.

Fix an arbitrary $\eta > 0$. We know from the potential problem (Lemma 2[17]) applied to the set K that the potential $f_{\bar{K}}$ generated by the charges $\{e_x; x \in \bar{K}\}$ is close to one in the surroundings of K, for a sufficiently large, connected set K. By (4.10) this is equivalent to saying that the charges have to vanish identically: $e_x \leqslant \eta$ for all x as long as K is large enough.

Using the FKG inequality,[20,21] one has the bound: for $x \in \partial K$

$$\langle S_x \mid E_K \rangle \geqslant \langle S_x \rangle(h) \tag{4.22}$$

where $\langle \cdot \rangle(h)$ is the measure obtained by setting $S_x = h$ for all $x \in K$, and conditioned on $S_x < h$, for all x. Explicitly,

$$\langle \cdot \rangle(h) \equiv \frac{1}{N} \int P_h(dS_{\partial K}) \cdot \prod_{\partial K} I(S_y < h)$$

with

$$P_h(dS_{\partial K}) \equiv \frac{1}{N} dS_{\partial K} \exp\left[-\frac{1}{2} \sum a_{xy} S_x S_y - \frac{1}{2} \sum (S_x - h)^2 \right] \tag{4.23}$$

i.e., the original (Gaussian) measure, but where $S_z = h$ for all $z \in K$. Performing a change of variables $S'_y = h - S_y$ in (4.22) and using (4.23) and (4.21) with $k = -h$, we get

$$\begin{aligned} \langle S_x \mid E_K \rangle &\geqslant h - \frac{1}{N} \int P_0(dS_{\partial K})\, S_x \prod_{\partial K} I(S_y > 0) \exp\left(\sum_{\partial K} h_y S_y \right) \\ &\geqslant h - \frac{1}{N} \int P_0(dS_{\partial K})\, S_x \prod_{\partial K} I(S_y > 0) \exp\left(\sum_{\partial K} h\eta S_y \right) \end{aligned} \tag{4.24}$$

where we used the FKG inequality again and $e_x < \eta$. The presence of the positive magnetic field η (which is small) comes from the coupling with infinity, which was at zero potential. If η were zero, then the second term to the right of the inequality (4.24) would just be the average value of S_x for a site x at the boundary of a set K where all the spins are frozen at $S_z = 0$, $z \in K$, with respect to the Gaussian measure perturbed by the $\prod I(S_y > 0)$ factor.

Due to the presence of the mass (and since a_{xy} is positive definite), we are in a position to apply Ruelle's superstability estimate[9] with interaction

$$U(S_{\partial K}) \equiv \frac{1}{2} \sum a_{xy} S_x S_y + \frac{1}{2} \sum S_x^2 \geqslant \frac{1}{2} \sum S_x^2$$

and single spin measure

$$\lambda(dS) \equiv e^{h\eta S} I(S>0)\, dS$$

Combining the Schwartz inequality

$$\frac{1}{N} \int P_0(dS_{\partial K})\, S_x \prod_{\partial K} \lambda(dS_y) \leqslant \left[\frac{1}{N} \int P_0(dS_{\partial K})\, S_x^2 \prod_{\partial K} \lambda(dS_y) \right]^{1/2}$$

with (4.24), we get as final bound (with const $<\infty$, and $A>0$, uniform in K)

$$\begin{aligned} \langle S_x \mid E_K \rangle &\geqslant h - \text{const} \cdot \left[\int \lambda(dS)\, S^2 \exp^{(-AS^2)} \right]^{1/2} \\ &\geqslant h - \text{const} \cdot O(1/\sqrt{A}) \\ &\geqslant c \end{aligned}$$

for some $h \geqslant c > 0$ if h is large enough and η small enough. The bound is uniform in $x \in \partial K$, $K \in F_V$ for all V large enough.

Conclusion of Proof of Theorem 3. Lemma 2(a) says that $\langle S_x \mid E_K \rangle$ is a harmonic function in $Z^d \backslash \bar{K}$. Lemma 3 says that this function is larger than a strictly positive constant c for all $x \in \bar{K}$, for h large enough, and zero at infinity. Hence, by the principle of domination[17] for harmonic functions

$$\langle S_x \mid E_K \rangle \geqslant c f_{\bar{K}}(x), \qquad \text{for all} \quad x \in Z^d \tag{4.25}$$

For $d=3$ we can apply Lemma 2(b) and combine it with (4.25): there is a constant $\tilde{\mu} > 0$ such that for all Λ, we can choose $V = V(\Lambda)$ large enough such that

$$\frac{1}{|\Lambda|} \langle S_\Lambda \mid E_K \rangle \geqslant \tilde{\mu}, \qquad \text{for all} \quad K \in F_V \tag{4.26}$$

By Lemma 1,

$$\langle S_\Lambda^2 \rangle \geqslant \langle S_\Lambda^2 I(C_V) \rangle = \sum_{K \in F_V} \langle S_\Lambda^2 I(E_K) \rangle = \sum_{K \in F_V} \langle S_\Lambda^2 \mid E_K \rangle \operatorname{Prob}(E_K)$$

and by the Schwartz inequality,

$$\geqslant \sum_{K \in F_V} \langle S_\Lambda \mid E_K \rangle^2 \operatorname{Prob}(E_K) \geqslant \sum_{K \in F_V} \tilde{\mu}^2 |\Lambda|^2 \operatorname{Prob}(E_K)$$

where we used (4.26) for the last inequality. Now, by Lemma 1 again,

$$=\tilde{\mu}^2|A|^2\,\mathrm{Prob}(C_V)\geqslant\tilde{\mu}^2|A|^2\,P_\infty(h)$$

Since this chain of inequalities holds for all A and $\langle[(1/|A|)\,S_A]^2\rangle\to 0$, for $A\uparrow Z^3$, we obtain that $P_\infty(h)=0$.

This completes the proof.

5. CONCLUDING REMARKS

5.1. Extensions to Other Dimensions

There exists a more general version of Wiener's test,[18] valid for all $d>2$, which is obtained by replacing $T[A]$ in (4.16) by

$$T_d[A]\equiv\sum_1^\infty 2^{n(2-d)}\,\mathrm{Cap}(A_n)$$

The main reason our result is restricted to $d=3$ is that we use, in Lemma 2, the fact that $T_3[A]=\infty$ for any infinite, connected set A. This fails in $d=4$, as can be seen explicitly by considering the set $A=$ a lattice axis. In $d=4$, only "higher dimensional" sets (like a plane) will have $T_d[A]=\infty$. However, for sets "like" a line, we can exclude percolation by using the bound (3.15). More precisely, if we fix some $K<\infty$, then there will be an h so that there is no infinite, connected set C on which $S_x>h$, such that $\chi_C\leqslant K$. Infinite "one-dimensional" sets in $d=4$ will satisfy this last condition. However, there are sets C with infinite C-susceptibility χ_C and $T(C)<\infty$, so that we cannot exclude percolation.

5.2. The Voter Model

In Ref. 19 the threshold percolation density p_c was investigated for another strongly correlated lattice system, the Voter model. As in the massless harmonic crystal, the pair correlation function of the system decays in three dimensions as $1/|x|$. However, there is an additional complication: the system does not satisfy the Markov property. One can no longer rely on methods of equilibrium statistical mechanics (the probability measure is characterized entirely by being stationary with respect to a certain type of stochastic time evolution). Therefore, the arguments presented in our proofs do not work for this system. The numerical work in Ref. 19 suggests that there is a nontrivial percolation transition, with $p_c\approx 0.16$ in $d=3$.

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REFERENCES

1. M. Aizenman and D. J. Barsky, *Commun. Math. Phys.* **108**:489 (1987).
2. C. Fortuin and P. Kasteleyn, *Physica* **57**:536 (1972).
3. S. R. Broadbent and J. M. Hammersley, *Proc. Camb. Phil. Soc.* **53**:629 (1957).
4. H. Kesten, *Percolation Theory for Mathematicians* (Birkhauser, Boston, 1982).
5. D. Stauffer, *An Introduction to Percolation Theory* (Taylor and Francis, 1986).
6. L. Russo, *Commun. Math. Phys.* **67**:251 (1979).
7. A. L. R. Bug, S. A. Safran, G. S. Grest, and I. Webman, *Phys. Rev. Lett.* **55**:1896 (1985).
8. S. A. Molchanov and A. K. Stepanov, *Theor. Math. Phys.* **55**:478 (1983).
9. D. Ruelle, *Commun. Math. Phys.* **50**:189 (1976).
10. C. Newman, *Commun. Math. Phys.* **41**:1 (1975).
11. Yu. A. Rozanov, *Markov Random Fields* (Springer-Verlag, New York, 1982).
12. J. Bricmont, J. L. Lebowitz, and C. E. Pfister, *J. Stat. Phys.* **24**:269 (1981).
13. H. J. Brascamp, J. L. Lebowitz, and E. H. Lieb, *Bull. Int. Stat. Inst.* **46**, Invited Paper No. 62 (1975).
14. J. Fröhlich, B. Simon, and T. Spencer, *Commun. Math. Phys.* **50**:79 (1976).
15. J. P. Kahane, *Some Random Series of Functions* (Cambridge University Press, Cambridge, 1985).
16. J. L. Lebowitz and C. Maes, *J. Stat. Phys.* **46**:39 (1987).
17. F. Spitzer, *Principles of Random Walk* (Van Nostrand, Princeton, New Jersey, 1964).
18. K. Itô and H. J. McKean, Jr., *Ill. J. Math.* **4**:119 (1960).
19. J. L. Lebowitz and H. Saleur, *Physica* **138A**:194 (1986).
20. C. Fortuin, P. Kasteleyn, and J. Ginibre, *Commun. Math. Phys.* **22**:89 (1971).
21. G. Battle and L. Rosen, *J. Stat. Phys.* **22**:128 (1980).
22. J. Bricmont, J. R. Fontaine, and L. J. Landau, *Commun. Math. Phys.* **56**:281 (1977).
23. D. Ruelle, *Thermodynamic Formalism* (Addison-Wesley, Reading, Massachusetts, 1978).